

# Self-consistent inhomogeneous steady states in Hamiltonian mean field dynamics

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Long-lived *quasistationary states*, associated with stationary stable solutions of the Vlasov equation, are found in systems with long-range interactions. Studies of the relaxation time in a model of  $N$  globally coupled particles moving on a ring, the Hamiltonian Mean Field model (HMF), have shown that it diverges as  $N^\gamma$  for large  $N$ , with  $\gamma \simeq 1.7$  for some initial conditions with homogeneously distributed particles. We propose a method for identifying *exact inhomogeneous steady states* in the thermodynamic limit, based on analysing models of uncoupled particles moving in an external field. For the HMF model, we show numerically that the relaxation time of these states diverges with  $N$  with the exponent  $\gamma \simeq 1$ . The method, applicable to other models with globally coupled particles, also allows an exact evaluation of the stability limit of homogeneous steady states. In some cases it provides a good approximation for the correspondence between the initial condition and the final steady state.

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## I. INTRODUCTION

Systems with long-range interactions exhibit unusual thermodynamic and dynamical properties [1]. These systems are characterized by two-body interaction potentials which decay at large distances  $r$  as  $r^{-\alpha}$ , with  $\alpha$  smaller than the spatial dimension  $d$ . With such potentials these systems are not additive and, as a result, they display characteristic features such as ensemble inequivalence, negative specific heat, temperature jumps and broken ergodicity.

Studies of the dynamics of a class of models have shown that some initial macroscopic states display slow relaxation to equilibrium. Starting from these initial states the system evolves on a short timescale towards a state that is not the one predicted by equilibrium statistical mechanics. The lifetime of this *quasistationary state* (QSS) diverges algebraically with system size. For a finite system one eventually relaxes towards thermal equilibrium [2]. The full nonlinear dynamical relaxation process remains to be fully characterized.

A prototypical model for which this dynamical behavior has been explored in detail is the Hamiltonian Mean Field model (HMF)[3], which describes the motion on a ring of  $N$  fully coupled particles. Its Hamiltonian is

$$\mathcal{H} = \sum_{i=1}^N \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^N (1 - \cos(\theta_j - \theta_i)) \quad (1)$$

where  $-\pi < \theta_i \leq \pi$  is the position of the  $i$ -th particle and  $p_i$  is its conjugated momentum. The model has a second order phase transition at the energy  $u_c = \mathcal{H}/N = 3/4$  at which the order parameter  $\mathbf{m} = (m_x, m_y) =$

$\sum_i (\cos \theta_i, \sin \theta_i)/N$  changes from zero (above  $u_c$ ) to non zero (below  $u_c$ ). This order parameter is referred to as the magnetization vector.

For this mean-field model, as for other long-range interacting systems, the time evolution of the single-particle distribution function  $f(\theta, p, t)$  is conveniently studied in the large  $N$  limit by the Vlasov equation [4]

$$\frac{\partial f}{\partial t} + p \frac{\partial f}{\partial \theta} - \frac{\partial V[f](\theta, t)}{\partial \theta} \frac{\partial f}{\partial p} = 0, \quad (2)$$

where

$$V[f](\theta, t) = \iint d\theta' dp' f(\theta', p', t) (1 - \cos(\theta' - \theta)) \quad (3)$$

is the effective potential seen by the particles.

It has been proposed that linearly stable stationary solutions of the Vlasov equation are associated with QSS of the large but finite  $N$  system [2]. This has been discussed in detail for *homogeneous*,  $\theta$  independent, steady states where  $\mathbf{m} = 0$ . Studying *inhomogeneous* ( $\mathbf{m} \neq 0$ ) steady states in this class of models is less straightforward [5]. Exploring these stationary states, their stability and the relaxation process for quasistationarity is a challenging goal.

The stationary solutions  $f_S(\theta, p)$  of the Vlasov equation are functions of the integrals of motion and thus, in one dimension, they are generically of the form

$$f_S(\theta, p) = F(h(\theta, p)) \text{ with } h(\theta, p) = \frac{p^2}{2} + V[f_S](\theta). \quad (4)$$

Here  $h(\theta, p)$  is the single-particle energy. The function  $F$  has to satisfy a self-consistency condition obtained by

inserting (4) into (3). However, this is satisfied by a wide class of functions  $F$ , reflecting the fact that the Vlasov equation has many stationary states. In this regard, a widely used concept is that of BGK modes [4, 6]. An application of this method to the HMF model has been proposed in [7]. We will here propose a different approach.

In this paper we introduce a simple method for constructing inhomogeneous steady states by analyzing the dynamics of models of non-interacting particles moving in an external potential. Relaxation in such systems has already been considered in Refs. [8–10], however it has not been used to derive analytically inhomogeneous steady states of the interacting particles systems. A different approach yielding approximate inhomogeneous steady states in systems with long-range interactions has previously been introduced by Lynden-Bell [11] and subsequently applied in a variety of models [10, 12–14]. Our method yields exact steady states and enables one to evaluate their stability limit. It also provides a procedure to relate in an approximate way an initial distribution to the steady state to which it evolves. For the sake of clarity, the method is applied to the HMF model, although it could be readily formulated in a more general context.

The paper is organized as follows. In Section II we introduce the uncoupled particles model and we derive the invariant measure. Section III presents the application of our method to the HMF model. In Section IV we derive the exact stability limit of the homogeneous waterbag state. Section V is devoted to some conclusions.

## II. THE UNCOUPLED PARTICLES MODEL

In order to find the stationary states of the HMF model, we consider the dynamics of an ensemble of *uncoupled particles* moving in a fixed external field  $H$  pointing, say, in the  $x$ -direction. The energy of a single particle is given by

$$\epsilon(\theta, p) = \frac{p^2}{2} - H \cos \theta . \quad (5)$$

For an arbitrary function  $F(\epsilon(\theta, p))$  to be a steady state of the interacting model (1),  $H$  has to satisfy the self-consistency condition

$$H = m_x = \iint d\theta dp F(\epsilon(\theta, p)) \cos \theta \quad ; \quad m_y = 0 . \quad (6)$$

In the following we suppress the subscript  $x$ . Any  $F$  that satisfies these conditions yields an exact stationary solution of the HMF model. In order to relate an initial distribution to the steady state to which it evolves, we consider an initial distribution of particles  $f_0(\theta, p)$ . The dynamics of the uncoupled particles model (5) is such that particles in a given energy shell  $[\epsilon, \epsilon + d\epsilon]$  keep moving inside that shell, eventually reaching a homogeneous distribution within it. As a result, the system attains the

following steady state distribution

$$P(\theta, p) = \frac{\iint d\theta' dp' f_0(\theta', p') \delta(\epsilon(\theta', p') - \epsilon(\theta, p))}{\iint d\theta' dp' \delta(\epsilon(\theta', p') - \epsilon(\theta, p))} . \quad (7)$$

For simplicity, we present below the analysis for the often studied *waterbag* initial condition

$$f_0(\theta, p) = \begin{cases} (4\Delta\theta\Delta p)^{-1} , & \text{for } |\theta| \leq \Delta\theta \text{ and } |p| \leq \Delta p , \\ 0 , & \text{otherwise.} \end{cases} \quad (8)$$

In order to evaluate  $P(\theta, p)$ , it is convenient to first consider the energy distribution  $P_\epsilon(\epsilon)$ . For the waterbag initial state (8) it is given by

$$P_\epsilon(\epsilon) = \frac{1}{4\Delta\theta\Delta p} \int d\theta \int_{-\Delta p}^{\Delta p} dp \delta\left(\frac{p^2}{2} - H \cos \theta - \epsilon\right) . \quad (9)$$

Carrying out the integral over  $p$ , one gets

$$P_\epsilon(\epsilon) = \frac{1}{2\Delta\theta\Delta p} \int d\theta \frac{1}{\sqrt{2(\epsilon + H \cos \theta)}} , \quad (10)$$

for  $-H \leq \epsilon \leq \Delta p^2/2 - H \cos \Delta\theta$  and zero outside this range. The integration over  $\theta$  needs to be done in the domain enclosed by the waterbag initial condition, namely

$$0 \leq \epsilon + H \cos \theta \leq \frac{\Delta p^2}{2} , \quad (11)$$

Thus,

$$P_\epsilon(\epsilon) = \frac{\sqrt{2}}{2\Delta\theta\Delta p} \int_{\theta_1}^{\theta_0} d\theta \frac{1}{\sqrt{(\epsilon + H \cos \theta)}} \quad (12)$$

where  $\theta_0$  and  $\theta_1$  satisfy (see Fig. 1a)

$$\theta_0 = \begin{cases} \arccos(-\epsilon/H) , & \text{for } -H < \epsilon < -H \cos \Delta\theta \\ \Delta\theta , & \text{for } \epsilon \geq -H \cos \Delta\theta , \end{cases} \quad (13)$$

$$\theta_1 = \begin{cases} 0 , & \text{for } -H < \epsilon \leq \Delta p^2/2 - H \\ \arccos(\frac{\Delta p^2/2 - \epsilon}{H}) , & \text{for } \Delta p^2/2 - H \leq \epsilon < \Delta p^2/2 - H \cos \Delta\theta \\ \Delta\theta , & \text{for } \epsilon \geq \Delta p^2/2 - H \cos \Delta\theta . \end{cases} \quad (14)$$

In the steady state, the  $(\theta, p)$  distribution is such that, for any given energy, all the microstates corresponding to that energy are equally probable. The boundaries on  $(\theta, p)$  imposed by the initial waterbag are no longer valid. Thus, the steady state distribution  $P(\theta, p)$  may be expressed as

$$P(\theta, p) = \frac{1}{4\Delta\theta\Delta p} \frac{P_\epsilon(\epsilon(\theta, p))}{Q_\epsilon(\epsilon(\theta, p))} \equiv \bar{P}_\epsilon(\epsilon(\theta, p)) , \quad (15)$$

where  $Q_\epsilon(\epsilon(\theta, p))$  is given by  $P_\epsilon(\epsilon(\theta, p))$ , evaluated by Eq. (12), with  $\theta_1 = 0$  and  $\theta_0$  given in (13) with  $\Delta\theta$  replaced by  $\pi$  (see Fig. 1a).

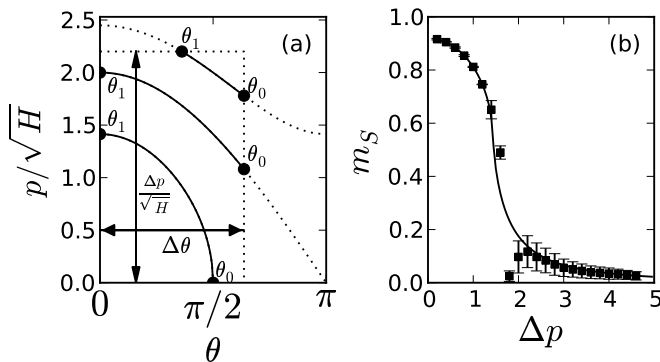


FIG. 1: a) Graphical representation of the computation of Eq. (12). The full lines represent the domain on which the integral is performed in order to compute  $P_\epsilon$  for values of the energy  $\epsilon = 0$ ,  $H$  and  $2H$ . The boundaries  $\theta_0$  and  $\theta_1$  are displayed for each energy. b) Magnetization in the steady state  $m_s$  vs.  $\Delta p$  for an initial waterbag state with  $\Delta\theta = 1$ . The full line is the theoretical prediction and the points are the results of numerical integrations of the Vlasov equation for the HMF model in Eqs. (2) and (3). Error bars quantify fluctuations of  $m_s$  in the steady state.

Integrating over  $p$  and using (5), it is straightforward to express, without any approximation, the marginal in  $\theta$  as

$$P_\theta(\theta, H) = \sqrt{2} \int_{-H \cos \theta}^{\infty} d\epsilon \frac{1}{\sqrt{(\epsilon + H \cos \theta)}} \bar{P}_\epsilon(\epsilon). \quad (16)$$

The interpretation of  $\bar{P}_\epsilon$  is straightforward. For any given energy  $\epsilon$ , it is proportional to the ratio of the time taken to cover the orbit in the  $(\theta, p)$ -plane confined by the waterbag domain to the time corresponding to the orbit unrestricted by the initial condition.  $\bar{P}_\epsilon$  vanishes for  $\epsilon < -H$ , becomes  $1/(4\Delta\theta\Delta p)$  in the interval  $-H < \epsilon < \epsilon^*$ , with  $\epsilon^* = \min(-H \cos \Delta\theta, \Delta p^2/2 - H)$ , and monotonously decreases to zero in  $\epsilon^* < \epsilon < \Delta p^2/2 - H \cos \Delta\theta$ .

### III. APPLICATION TO WATERBAG STATES OF THE HMF MODEL

We now use these results in order to probe the steady states of the HMF model, where the particles are now interacting. For the distribution (15) to be a stationary state of the corresponding HMF model one has to demand that the steady state magnetization of the model with non interacting particles satisfies  $H = m$  [15], where

$$m = \int_{-\pi}^{+\pi} d\theta P_\theta(\theta, H) \cos \theta. \quad (17)$$

The strategy we propose is to solve the above self-consistency equation in  $H$  and then substitute the selected value of  $H$  into Eq. (15) to get the steady state distributions, not only of the uncoupled system but also

of the HMF. This is the main result of this paper. In applying these results to the HMF model, it is assumed that the magnetization takes its steady state value instantaneously at  $t = 0$ . This is of course not correct since the magnetization evolves towards its steady state value on a finite time scale, during which the momentum distribution also changes. Thus the correspondence between the initial state and the steady state distribution is only approximate. This approximation works well for some initial conditions and not so well for others, as discussed below.

We proceed by solving the self-consistency equation (17). For all  $\Delta\theta < \pi$  and at small  $H$  the leading term in the marginal  $P_\theta(\theta, H)$  behaves like  $\sqrt{H}$ . This can be easily proven numerically but also analytically by splitting the integral in (16) in the three relevant domains of integration  $[-H \cos \theta, -H \cos \Delta\theta]$ ,  $[-H \cos \Delta\theta, \Delta p^2/2 - H]$ ,  $[\Delta p^2/2 - H, \Delta p^2/2 - H \cos \Delta\theta]$ . The integration on both the first and the second domain displays a  $\sqrt{H}$  behavior and the first integral can be explicitly computed, giving  $2\sqrt{2H(\cos \theta - \cos \Delta\theta)}$ . The third integral gives an order  $H^2$  contribution. When computing  $m$  in formula (17) the first and second integral contribute terms that are opposite in sign for large enough  $\Delta\theta$  and do not cancel each other. When  $\Delta\theta$  reaches  $\pi$  the  $\sqrt{H}$  terms cancel, giving rise in this limit to an order  $H$  global contribution (we give below explicitly the result for  $\Delta\theta = \pi$ ). Because of the  $\sqrt{H}$  behavior, the  $H = 0$  solution is always unstable. Therefore the magnetization is expected to be non zero at any value of  $\Delta p$  [13, 16]. The theoretical magnetic curve obtained by (16) and (17) is compared with the numerical solution of the Vlasov equation for  $\Delta\theta = 1$  in Fig. 1b. The agreement is quite good, considering that the theoretical prediction has no free parameters.

In Fig. 2 we compare the marginals in  $\theta$  and in  $p$  of the steady state distribution (15) with those obtained numerically by direct integration of the Vlasov equation. Apart from the oscillations observed in the tails, which are due to the filamentation of the initial waterbag distribution, the agreement is quite good.

Let us now consider initial homogeneous waterbag states, for which  $\Delta\theta = \pi$ . In analysing Eq. (17) we find that  $m = H = 0$  is a solution for any  $\Delta p$ . At low  $\Delta p$  there exists one additional solution with  $m \neq 0$ , while in a rather narrow intermediate range  $\Delta p^* < \Delta p < \bar{\Delta p}$  two solutions with nonzero magnetization are present. In Fig. 3a we plot  $m(H) - H$  as a function of  $H$  for  $\Delta p$  in the intermediate region where one gets three solutions of the self-consistency equation, two stable (S) and one unstable (U). In Fig. 3b we plot the non zero self-consistent steady value of the magnetization,  $m_s$ , versus  $\Delta p$ . In this case the theoretical prediction does not agree with the numerical simulations of the Vlasov equation. We argue that this is due to the fact that, during time evolution, there is a significant mixing of energy levels, due to strong variations of the effective field near a discontinuous first order transition [17].

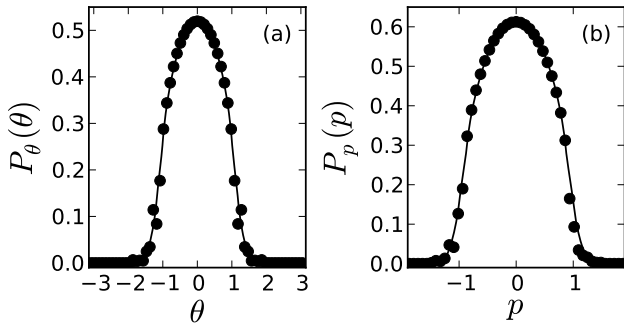


FIG. 2: Marginals in  $\theta$ ,  $P_\theta(\theta) = \int dp P(\theta, p)$ , (panel a) and in  $p$ ,  $P_p(p) = \int d\theta P(\theta, p)$ , (panel b) of the steady state distribution (15) for  $\Delta\theta = 1$  and  $\Delta p = 1$ . The full lines are the theoretical predictions and the points are the results of numerical integrations of the Vlasov equation for the HMF model in Eqs. (2) and (3).

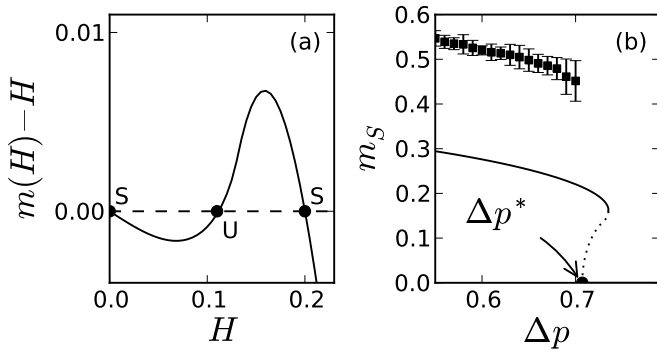


FIG. 3: a)  $m(H) - H$  vs.  $H$  for  $\Delta p = 0.72$  within the range of existence of three solutions  $1/\sqrt{2} = \Delta p^* < \Delta p < \bar{\Delta p} \simeq 0.735$ . Two stable fixed point (S) with negative slopes and an unstable one (U) with a positive slope. b) The non zero steady state magnetization  $m_S$  as a function of  $\Delta p$ . The dotted segment corresponds to unstable solutions and the full line to stable ones. The dot on the real axis marks the stability limit of the homogeneous state  $\Delta p^*$ . The points are results of numerical integrations of the Vlasov equation for the HMF model in Eqs. (2) and (3).

#### IV. STABILITY OF THE HOMOGENEOUS WATERBAG STATE

We now analyze the stability of the stationary solutions. For the zero magnetization state the stability limit is determined by the slope of  $m(H)$  at  $H = 0$ , which is obtained by taking the small  $H$  limit in the integral (16). We therefore expand  $P_\theta(\theta, H)$  in powers of  $H$  keeping only the linear term in  $H$  which is proportional to  $\cos\theta$ . This term is found to be  $H \cos\theta / (2\pi\Delta p^2)$ . It is obtained by carrying out the integration in the energy domain  $-H \cos\theta < \epsilon < \Delta p^2/2 - H$ , where  $\bar{P}_\epsilon(\epsilon)$  is constant. The stability threshold for the  $m = 0$  solution is thus  $\Delta p^* = 1/\sqrt{2}$ , as displayed in Fig.3b. For  $\Delta p > \Delta p^*$  the homogeneous state is stable and it becomes unsta-

ble below  $\Delta p^*$ . Using this value in the HMF model we get the energy  $(\Delta p^*)^2/6 + 1/2 = 7/12$ , which coincides with the threshold energy calculated using other methods [2, 3]. This analysis can be generalized to a generic initial momentum distribution.

The stable inhomogeneous solutions found for different values of  $\Delta p$  are exact stable stationary solutions of the Vlasov equation, but they evolve in time for finite  $N$ . Numerical studies of the time evolution of the magnetization are summarized in Fig. 4, where  $m(t)$  is displayed for several values of  $N$ . We obtain a data collapse when time is rescaled by  $1/N$ , implying that the relaxation time increases as  $N^\gamma$  with  $\gamma = 1$  [18, 19]. This is at variance with the relaxation of homogeneous states where  $\gamma \simeq 1.7$  [2, 20, 21].

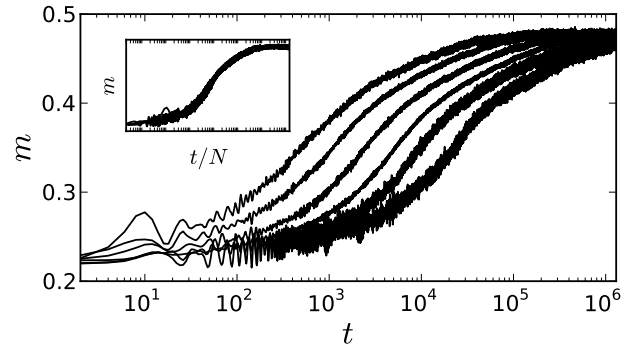


FIG. 4: Magnetization vs. time of the HMF model starting with the stable inhomogeneous steady state at  $\Delta p = 0.7$  for  $N = 2^{10}(100), 2^{11}(100), 2^{12}(50), 2^{13}(50), 2^{14}(20), 2^{15}(20)$  from left to right (in parenthesis the number of realisations). In the inset the same data are plotted as a function of  $t/N$ .

#### V. CONCLUSIONS

In summary, our approach allows one to derive exact inhomogeneous steady states. It gives approximate correspondence between initial and final states, which, for the HMF model, yields good results for initial inhomogeneous waterbags. Moreover, one obtains the exact stability limit of the zero magnetization state, since the waterbag initial condition is in itself a steady state with zero magnetization and the dynamics of each particle is therefore exactly described by the independent particles model (5).

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